

De Rham theorem for Whitney functions

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Abstract

Let M be a real analytic manifold, F a bounded complex of constructible sheaves. We show that the Whitney-de Rham complex associated to F is quasi-isomorphic to F .

1 Introduction

Let M be a real analytic manifold, by Poincaré Lemma, it is well-known that the de Rham complex with \mathcal{C}^∞ -coefficients over M is isomorphic to \mathbb{C}_M . In this paper we show that \mathbb{C}_M is isomorphic to the de Rham complex with Whitney coefficients over M on the associated subanalytic site. Then, using sheaf theoretical arguments we can easily prove that, given a bounded complex of constructible sheaves F , the Whitney-de Rham complex associated to F is quasi-isomorphic to F . As a corollary we obtain a theorem of [4]. (Another proof was given in [5] using deep results on \mathcal{D} -modules.) We also obtain a de Rham theorem for Schwartz functions on open subanalytic subsets.

Our proof uses these three known facts:

- There exists a site, the so called subanalytic site, where Whitney functions form a sheaf.
- Locally on the site, the sections of this sheaf are nothing but \mathcal{C}^∞ -functions with bounded derivatives.
- The homotopy axiom holds for de Rham cohomology with bounded derivatives.

Let us detail our arguments.

It is well known that Whitney \mathcal{C}^∞ -functions on the real analytic manifold M do not satisfy the gluing conditions on open coverings. Hence, they do

not form a sheaf. However, following [8], one can overcome this problem by associating to M a Grothendieck topology by choosing as open subsets subanalytic open subsets and as coverings locally finite ones. This is the subanalytic site M_{sa} , and here Whitney \mathcal{C}^∞ -functions form a (subanalytic) sheaf denoted $\mathcal{C}_M^{\infty,w}$.

On M_{sa} , one can also define the (subanalytic) sheaf of \mathcal{C}^∞ -functions with bounded derivatives on bounded open subanalytic subsets, and prove that the associated de Rham complex is quasi-isomorphic to the constant sheaf on M .

\mathcal{C}^∞ -functions on U with bounded derivatives coincide with Whitney \mathcal{C}^∞ -functions on \overline{U} when U is 1-regular (see [12]). Thanks to a decomposition result of [10] (which uses techniques developed in [9]) one can prove that locally on M_{sa} the sheaf of bounded \mathcal{C}^∞ -functions coincide with $\mathcal{C}_M^{\infty,w}$.

Moreover, applying some (subanalytic) sheaf theoretical techniques one can link the de Rham complex with coefficients in $\mathcal{C}_M^{\infty,w}$ with the one with Whitney coefficients on any closed subanalytic subset Z of M . This is because, basically, $\mathrm{RHom}(D'\mathbb{C}_Z, \mathcal{C}_M^{\infty,w})$ is quasi-isomorphic to the space of Whitney \mathcal{C}^∞ -functions on Z . Here \mathbb{C}_Z denotes the constant sheaf on Z and $D'\mathbb{C}_Z$ its dual $R\mathcal{H}om(\mathbb{C}_Z, \mathbb{C}_M)$. This is done thanks to the Whitney tensor product introduced by Kashiwara-Schapira in [7]. When $Z = \overline{U}$, with U open 1-regular relatively compact and such that $D'\mathbb{C}_U \simeq \mathbb{C}_Z$, we obtain that $\mathrm{RHom}(D'\mathbb{C}_Z, \mathcal{C}_M^{\infty,w})$ is the space of \mathcal{C}^∞ -functions on U with bounded derivatives.

Thanks to these considerations, one can reduce the isomorphism between the cohomology of Z and the Whitney de Rham complex to the quasi-isomorphism between \mathbb{C}_M and the de Rham complex with coefficients in $\mathcal{C}_M^{\infty,w}$, which is locally the homotopy axiom for de Rham cohomology with bounded derivatives and can be obtained as in [3].

The statement still holds if we replace \mathbb{C}_Z with a bounded complex of constructible sheaves F and Whitney \mathcal{C}^∞ -functions with the Whitney tensor product $F \overset{w}{\otimes} \mathcal{C}_M^\infty$. This allows to consider different kinds of de Rham theorems, as the one for Schwartz functions when $F = \mathbb{C}_U$, U open subanalytic.

2 Subanalytic sheaves

The following results on subanalytic sheaves are extracted from [8] (see also [11]). We refer to [6] for classical sheaf theory.

Let X be a real analytic manifold. Denote by $\text{Op}(X_{sa})$ the category of open subanalytic subsets of X . One endows $\text{Op}(X_{sa})$ with the following topology: $S \subset \text{Op}(X_{sa})$ is a covering of $U \in \text{Op}(X_{sa})$ if for any compact K of X there exists a finite subset $S_0 \subset S$ such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site.

Let $\text{Mod}(\mathbb{C}_{X_{sa}})$ (resp. $D^b(\mathbb{C}_{X_{sa}})$) denote the category of sheaves (resp. bounded complexes of sheaves) of \mathbb{C} -vector spaces on X_{sa} and let $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ (resp. $D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$) be the abelian category of \mathbb{R} -constructible sheaves (resp. bounded complexes of sheaves with \mathbb{R} -constructible cohomology) on X .

We denote by $\rho : X \rightarrow X_{sa}$ the natural morphism of sites. We have functors

$$\begin{aligned} \rho_* : \text{Mod}(k_X) &\rightarrow \text{Mod}(k_{X_{sa}}) \\ \rho^{-1} : \text{Mod}(k_{X_{sa}}) &\rightarrow \text{Mod}(k_X) \\ \rho_! : \text{Mod}(k_X) &\rightarrow \text{Mod}(k_{X_{sa}}) \end{aligned}$$

The functors ρ^{-1} and ρ_* are the functors of inverse image and direct image associated to ρ . The functor $\rho_!$ is left adjoint to ρ^{-1} . It is fully faithful and exact. If $F \in \text{Mod}(\mathbb{C}_{X_{sa}})$, $\rho_! F$ is the sheaf associated to the presheaf $U \mapsto \Gamma(\overline{U}; F)$. The functor ρ_* is fully faithful and exact on $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and we identify $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ with its image in $\text{Mod}(\mathbb{C}_{X_{sa}})$ by ρ_* .

3 Whitney tensor product

Let M be a real analytic manifold. We denote by \mathcal{C}_M^∞ (resp. \mathcal{A}_M) the sheaf of \mathcal{C}^∞ -functions (resp. analytic functions) on M and by \mathcal{D}_M the sheaf of differential operators on M with analytic coefficients. We denote by Ω_M^p the sheaf of p -differential forms with coefficients in \mathcal{A}_M and by Ω_M^\bullet the complex

$$0 \rightarrow \Omega_M^0 \rightarrow \dots \rightarrow \Omega_M^{\dim M} \rightarrow 0$$

We denote by $\text{Mod}(\mathcal{D}_M)$ (resp. $D^b(\mathcal{D}_M)$) the category (resp. bounded derived category) of sheaves of \mathcal{D}_M -modules. References are made to [7] for a complete exposition on formal cohomology.

Definition 3.1 Let Z be a closed subset of M . We denote by $\mathcal{I}_{M,Z}^\infty$ the sheaf of \mathcal{C}^∞ -functions on M vanishing up to infinite order on Z .

Definition 3.2 A Whitney function on a closed subset Z of M is an indexed family $F = (F^k)_{k \in \mathbb{N}^n}$ consisting of continuous functions on Z such that $\forall m \in \mathbb{N}, \forall k \in \mathbb{N}^n, |k| \leq m, \forall x \in Z, \forall \varepsilon > 0$ there exists a neighborhood U of x such that $\forall y, z \in U \cap Z$

$$\left| F^k(z) - \sum_{|j+k| \leq m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leq \varepsilon d(y, z)^{m-|k|}.$$

We denote by $W_{M,Z}^\infty$ the space of Whitney \mathcal{C}^∞ -functions on Z . We denote by $\mathcal{W}_{M,Z}^\infty$ the sheaf $U \mapsto W_{U, U \cap Z}^\infty$.

In [7] the authors defined the functor

$$\cdot \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty : \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \rightarrow \text{Mod}(\mathcal{D}_M)$$

in the following way: let U be a subanalytic open subset of M and $Z = M \setminus U$. Then $\mathbb{C}_U \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty = \mathcal{I}_{M,Z}^\infty$, and $\mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty = \mathcal{W}_{M,Z}^\infty$. This functor is exact and extends as a functor in the derived category, from $D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$ to $D^b(\mathcal{D}_M)$. Moreover the sheaf $F \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty$ is soft for any \mathbb{R} -constructible sheaf F .

Let $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$, one sets

$$F \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet} := F \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty \underset{\mathcal{A}_M}{\otimes} \Omega_M^\bullet.$$

4 The sheaf of Whitney functions

Here we recall definition and some properties of the subanalytic sheaf of Whitney functions. References are made to [8, 11].

Definition 4.1 One denotes by $\mathcal{C}_M^{\infty, \text{w}}$ the sheaf on M_{sa} of Whitney \mathcal{C}^∞ -functions defined as follows:

$$U \mapsto \Gamma(M; H^0 D'(\mathbb{C}_U) \overset{\text{w}}{\otimes} \mathcal{C}_M^\infty),$$

where $D'(\cdot) = R\mathcal{H}om(\cdot, \mathbb{C}_M)$.

Remark that $\Gamma(U, \mathcal{C}_M^{\infty, w})$ is a $\Gamma(\overline{U}; \mathcal{D}_M)$ -module for each $U \in \text{Op}(M_{sa})$, this implies that $\mathcal{C}_M^{\infty, w}$ has a structure of $\rho_! \mathcal{D}_M$ -module. We have the following result. For each $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$ one has the isomorphism

$$\rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty, w}) \simeq F \overset{w}{\otimes} \mathcal{C}_M^{\infty}.$$

Consider the complex Ω_M^\bullet of differential forms with analytic coefficients and set

$$\mathcal{C}_M^{\infty, w, \bullet} := \mathcal{C}_M^{\infty, w} \underset{\rho_! \mathcal{A}_M}{\otimes} \rho_! \Omega_M^\bullet.$$

One has the quasi-isomorphisms

$$\rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty, w, \bullet}) \simeq \rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty, w}) \underset{\mathcal{A}_M}{\otimes} \Omega_M^\bullet \simeq F \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet}$$

where the first quasi-isomorphism follows from [11, Proposition 2.2.3] and the fact that $\rho^{-1} \circ \rho_! \simeq \text{id}$.

Let us consider a locally cohomologically trivial (l.c.t.) subanalytic open subset, i.e. $U \in \text{Op}(M_{sa})$ satisfying $D'\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$ and $D'\mathbb{C}_{\overline{U}} \simeq \mathbb{C}_U$. Then

$$R\Gamma(U; \mathcal{C}_M^{\infty, w}) \simeq R\Gamma(M; \mathbb{C}_{\overline{U}} \overset{w}{\otimes} \mathcal{C}_M^{\infty})$$

is concentrated in degree zero. It means that $\mathcal{C}_M^{\infty, w}$ is the sheaf associated to the presheaf $U \mapsto \{\text{Whitney functions on } \overline{U}\}$. Indeed l.c.t. open subanalytic subsets form a basis for the topology of M_{sa} (i.e. every locally compact subanalytic open subset has a finite cover consisting of l.c.t. subanalytic open subsets).

5 Proof of the Theorem

Lemma 5.1 *There is the following isomorphism in $D^b(\mathbb{C}_{M_{sa}})$*

$$\mathbb{C}_M \simeq \mathcal{C}_M^{\infty, w, \bullet}.$$

Proof. It follows from [10, Theorem 0.3] that every relatively compact subanalytic open subset has a finite cover consisting of contractible 1-regular open subanalytic subsets. A result of [12] asserts that on 1-regular open subsets Whitney functions are bounded \mathcal{C}^∞ -functions with bounded derivatives. Hence we are reduced to prove the isomorphism in $D^b(\mathbb{C}_{M_{sa}})$

$$R\Gamma(U; \mathbb{C}_M) \simeq \Gamma(U; \mathcal{C}_M^{\infty, w, \bullet}).$$

Since U is contractible the left hand side is concentrated in degree 0 and $\Gamma(U; \mathbb{C}_M) \simeq \mathbb{C}$. We are reduced to prove that the right hand side is concentrated in degree zero and $H^0(U; \mathcal{C}_M^{\infty, w, \bullet}) \simeq \mathbb{C}$. This is nothing but the homotopy axiom for de Rham cohomology with bounded derivatives, which can be obtained in the same way as the classical one, see [3, Corollary 4.1.2] (one just checks that the homotopy operator K such that $f^* - g^* = K \circ d + d \circ K$ preserves bounded derivatives). \square

Theorem 5.2 *Let $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$. There is the following isomorphism in $D^b(\mathbb{C}_M)$*

$$F \simeq F \otimes_M^w \mathcal{C}_M^{\infty, \bullet}.$$

Proof. We have the chain of isomorphism in $D^b(\mathbb{C}_M)$

$$\begin{aligned} F \otimes_M^w \mathcal{C}_M^{\infty, \bullet} &\simeq \rho^{-1} R\mathcal{H}om(D'F, \mathcal{C}_M^{\infty, w, \bullet}) \\ &\simeq R\mathcal{H}om(D'F, \mathbb{C}_M) \\ &\simeq F, \end{aligned}$$

where the second isomorphism follows from Lemma 5.1 and the last one from the fact that $D'D'F \simeq F$ if $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M)$. \square

Corollary 5.3 *Let Z be a closed subanalytic subset of M . There is the following isomorphism in $D^b(\mathbb{C}_M)$*

$$\mathbb{C}_Z \simeq \mathcal{W}_{M,Z}^{\infty, \bullet}$$

where $\mathcal{W}_{M,Z}^{\infty, \bullet}$ denotes the de Rham complex with coefficients in $\mathcal{W}_{M,Z}^{\infty, \bullet}$, i.e. the Whitney-de Rham complex is isomorphic to \mathbb{C}_Z .

Proof. Set $F = \mathbb{C}_Z$ in Theorem 5.2. \square

Corollary 5.4 *Let Z be a closed subanalytic subset of M . There is the following isomorphism in $D^b(\mathbb{C})$*

$$R\Gamma(Z; \mathbb{C}_Z) \simeq R\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet}),$$

hence the cohomology $H^\bullet(Z)$ of Z is isomorphic to the cohomology the de Rham complex with Whitney coefficients on Z .

Proof. We have the following chain of isomorphisms in $D^b(\mathbb{C})$

$$R\Gamma(Z; \mathbb{C}_Z) \simeq R\Gamma(M; \mathbb{C}_Z) \simeq R\Gamma(M; \mathcal{W}_{M,Z}^{\infty, \bullet}),$$

where the first isomorphism follows from the definition of direct image and the second one follows applying $R\Gamma(M; \cdot)$ to Corollary 5.3. \square

Corollary 5.5 *Let U be an open subanalytic subset of M . There is the following isomorphism in $D^b(\mathbb{C})$*

$$R\Gamma_c(U; \mathbb{C}_U) \simeq R\Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet}),$$

where $\mathcal{I}_{M, M \setminus U}^{\infty, \bullet}$ denotes the de Rham complex with coefficients in $\mathcal{I}_{M, M \setminus U}^{\infty, \bullet}$, hence the cohomology with compact support $H_c^\bullet(U)$ of U is isomorphic to the cohomology with compact support of the de Rham complex with \mathcal{C}^∞ -coefficients on M vanishing with all their derivatives outside U .

Proof. Set $F = \mathbb{C}_U$ in Theorem 5.2. Then we have the following chain of isomorphisms in $D^b(\mathbb{C})$

$$R\Gamma_c(U; \mathbb{C}_U) \simeq R\Gamma_c(M; \mathbb{C}_U) \simeq R\Gamma_c(M; \mathcal{I}_{M, M \setminus U}^{\infty, \bullet}),$$

where the first isomorphism follows from the definition of proper direct image and the second one follows applying $R\Gamma_c(M; \cdot)$ to Theorem 5.2. \square

Remark 5.6 *As a consequence of Corollary 5.5, we obtain a result of [1]: the cohomology of the de Rham complex with Schwartz coefficients $H^\bullet DR_S(M)$ of a Nash manifold M is isomorphic to the compact support cohomology $H_c^\bullet(M)$ of M . We first prove it for \mathbb{R}^n . We can see \mathbb{R}^n as an open subset of \mathbb{S}^n . Let U be an open semialgebraic subset of \mathbb{R}^n . Then*

$$H_c^\bullet(U) \simeq H^\bullet(\mathbb{S}^n; \mathcal{I}_{\mathbb{S}^n, \mathbb{S}^n \setminus U}^{\infty, \bullet}) \simeq H^\bullet DR_S(U).$$

Using the fact that a Nash manifold has a finite cover consisting of open submanifolds Nash diffeomorphic to \mathbb{R}^n one obtains the result.

6 A CDGA quasi-isomorphism

Let Z be a closed subanalytic subset of M and consider the de Rham complex of sheaves

$$\mathbb{C}_Z \otimes^{\mathbf{w}} \mathcal{C}_M^{\infty, \bullet}.$$

Let us consider a resolution of Z

$$0 \rightarrow \oplus \mathbb{C}_{U_1} \rightarrow \cdots \rightarrow \oplus \mathbb{C}_{U_k} \rightarrow \mathbb{C}_Z \rightarrow 0$$

made by subanalytic open subsets (we may also assume they are contractible and locally cohomologically trivial), then $\mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}$ is defined by a sequence of chain complexes whose entries are of this kind

$$\mathbb{C}_V \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet},$$

with V locally cohomologically trivial. Now, by definition of $\mathcal{C}_M^{\infty, \text{w}}$, we have $\mathbb{C}_V \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty} \simeq \rho^{-1} \mathcal{H}om(\mathbb{C}_{\overline{V}}, \mathcal{C}_M^{\infty, \text{w}})$ when V is locally cohomologically trivial. It follows from Lemma 5.1 that the chain complex of (subanalytic) sheaves

$$\mathbb{C}_M \rightarrow \mathcal{C}_M^{\infty, \text{w}, \bullet}$$

is exact. Moreover \mathbb{C}_M and $\mathcal{C}_M^{\infty, \text{w}}$ are acyclic with respect to $\mathcal{H}om(\mathbb{C}_{\overline{V}}, \cdot)$ (V locally cohomologically trivial) and ρ^{-1} is exact, hence the chain complex

$$\mathbb{C}_V \simeq \mathcal{H}om(\mathbb{C}_{\overline{V}}, \mathbb{C}_M) \rightarrow \rho^{-1} \mathcal{H}om(\mathbb{C}_{\overline{V}}, \mathcal{C}_M^{\infty, \text{w}, \bullet}) \simeq \mathbb{C}_V \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}$$

is exact. This implies that the chain complex

$$\mathbb{C}_Z \rightarrow \mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}$$

is exact as well. Let us consider the exact sequence of chain complexes

$$0 \rightarrow \mathbb{C}_U \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet} \rightarrow \mathcal{C}_M^{\infty, \bullet} \rightarrow \mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet} \rightarrow 0$$

and apply the functor $\Gamma(W; \cdot)$. Being the Whitney tensor product a soft sheaf, we get an exact sequence of chain complexes

$$0 \rightarrow \Gamma(W; \mathbb{C}_U \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}) \rightarrow \Gamma(W; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \Gamma(W; \mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}) \rightarrow 0.$$

Remark that J is a CDGA (commutative differential graded algebra) morphism. We have

$$\begin{aligned} \Gamma(W; \mathbb{C}_U \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(W; \mathbb{C}_U), \\ \Gamma(W; \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(W; \mathbb{C}_M), \\ \Gamma(W; \mathbb{C}_Z \overset{\text{w}}{\otimes} \mathcal{C}_M^{\infty, \bullet}) &\simeq R\Gamma(W; \mathbb{C}_Z). \end{aligned}$$

Suppose that Z is a deformation retract of W . Then

$$\begin{aligned} R\Gamma(W; \mathbb{C}_U) &\simeq 0, \\ R\Gamma(W; \mathbb{C}_M) &\simeq R\Gamma(Z, \mathbb{C}_Z), \\ R\Gamma(W; \mathbb{C}_Z) &\simeq R\Gamma(Z; \mathbb{C}_Z), \end{aligned}$$

Hence for such a W

$$\Gamma(W; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \Gamma(W; \mathbb{C}_Z \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet}).$$

is a quasi-isomorphism of CDGA.

Remark 6.1 *Since every neighborhood of a subanalytic closed subset Z of M contains a neighborhood W which has a deformation retract to Z , we get a quasi-isomorphism of CDGA*

$$\Gamma(M; \mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet}) \simeq \varinjlim_W \Gamma(W; \mathcal{C}_M^{\infty, \bullet}) \xrightarrow{J} \varinjlim_W \Gamma(W; \mathbb{C}_Z \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet}) \simeq \Gamma(M; \mathbb{C}_Z \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet})$$

where W ranges through the family of neighborhoods having a deformation retract to Z . Thanks to this fact one can easily prove the quasi-isomorphism of sheaves of CDGA

$$\mathbb{C}_Z \otimes \mathcal{C}_M^{\infty, \bullet} \xrightarrow{\sim} \mathbb{C}_Z \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet}.$$

Remark 6.2 *Thanks to the exactness of $\cdot \otimes \mathcal{C}_M^{\infty}$ and $\cdot \overset{w}{\otimes} \mathcal{C}_M^{\infty}$, one can extend the quasi-isomorphism of Remark 6.1 to the case of an open subanalytic subset U and then, inductively, to the case of a \mathbb{R} -constructible sheaf F . Namely, we have a quasi-isomorphism of sheaves of CDGA*

$$F \otimes \mathcal{C}_M^{\infty, \bullet} \xrightarrow{\sim} F \overset{w}{\otimes} \mathcal{C}_M^{\infty, \bullet}.$$

Remark 6.3 *As a further application, one can extend the results of [2] (as pointed out by the authors in the introduction) to the case of subanalytic sets. One first remarks (following their notations) that the Whitney-de Rham cohomology of Z is isomorphic as a CGA to its singular cohomology with real coefficients:*

$$H^\bullet(Z, \mathbb{R}) \simeq H_W^\bullet(Z),$$

and these isomorphisms are compatible with the CGA structures. Then, when the set Z is simply connected, one can prove that the de Rham complex with Whitney coefficients $\Omega_W^\bullet(Z)$ determines the real homotopy type of Z . This also implies that the Hochschild homology of the differential graded algebra $\Omega_W^\bullet(Z)$ is isomorphic to the cohomology of the free loop space $\mathcal{L}Z$.

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